

Cohomology of Kähler Manifolds with $\mathbf{c}_1 = 0$

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To Professor Calabi on his 70th birthday

Introduction

A more informal title of this report might be ‘Betti, Hodge and Chern numbers’, without an implied order of preference. To begin with, let M be a compact connected Kähler surface, so that the real dimension of M is 4. The Betti numbers of M are given in terms of Hodge numbers by the usual relations

$$\begin{aligned} b_1 &= h^{1,0} + h^{0,1} = 2h^{0,1} \\ b_2 &= h^{2,0} + h^{1,1} + h^{0,2} = h^{1,1} + 2h^{0,2}. \end{aligned}$$

Noether’s formula states that

$$\left\langle \frac{1}{12}(\mathbf{c}_1^2 + \mathbf{c}_2), [M] \right\rangle = 1 - h^{0,1} + h^{0,2},$$

and the left-hand side is by definition the Todd genus of M . Moreover, the Euler characteristic $e(M)$ is equal to either side of the equation

$$\langle \mathbf{c}_2, [M] \rangle = 2 - 2b_1 + b_2.$$

If the first Chern class \mathbf{c}_1 of M vanishes over \mathbb{Z} , then the canonical bundle of M is trivial, and $h^{0,2} = 1$. As a consequence, we obtain the formula

$$0 = 22 - 4b_1 - b_2. \tag{1}$$

In these circumstances, the classification of complex surfaces implies that M must in fact be a torus T ($b_1 = 4$, $b_2 = 6$) or a K3 surface K ($b_1 = 0$, $b_2 = 22$). We may construct K , at least up to diffeomorphism, by resolving the 16 singular points of the orbifold T/\mathbb{Z}_2 , where \mathbb{Z}_2 is generated by the mapping $x \mapsto -x$. Observe that the right-hand side of (1) assumes the value 16 for T/\mathbb{Z}_2 ($b_1 = 0$, $b_2 = 6$); in fact, we shall see that the formula has a significance which transcends the examples.

We move on to consider the case in which M is now a Kähler manifold of *complex* dimension 4. The introduction to [20] contains a formula which (after slight rearrangement) bears a striking similarity to (1). Namely, if χ^p denotes the alternating sum $\sum_{q=0}^4 (-1)^q h^{p,q}$ of Hodge numbers of M then

$$\langle \mathbf{c}_1 \mathbf{c}_3, [M] \rangle = 22\chi^0 - 4\chi^1 - \chi^2. \tag{2}$$

This equation is generalized by Theorem 2, which asserts that on any almost complex n -dimensional manifold the characteristic number $\langle \mathbf{c}_1 \mathbf{c}_{n-1}, [M] \rangle$ can be expressed in terms of the integers χ^p ,

$0 \leq p \leq [n/2]$. This is a simple corollary of the Riemann-Roch theorem that was derived in [40] (an expanded version of this article) by differentiating a suitable K-theoretical expression. The author subsequently discovered that the corollary had been used earlier by Narasimhan and Ramanan [32], in a context discussed briefly at the end of Section 3. In any case, we deduce below that if $n = 2m$ is even and $\mathbf{c}_1 = 0$ then the ‘mirror symmetry’

$$h^{p,q} \longleftrightarrow h^{n-p,q}, \quad 0 \leq p, q \leq n, \quad (3)$$

reverses the sign of a certain linear combination Φ of Betti numbers, defined in Theorem 1. This complements the obvious fact that the operation (3) acts as $(-1)^n$ on the Euler characteristic e , and adds weight to the view that (3) has some significance in higher dimensions.

Compact hyper-Kähler manifolds constitute a class of Kähler manifolds with $\mathbf{c}_1 = 0$ and whose Hodge numbers are invariant by the symmetry (3). It follows from above that their Betti numbers are subject to the equation $\Phi = 0$, of which (1) is a special case. This provides an effective means to prove the non-existence of hyper-Kähler metrics on particular manifolds. The name ‘hyper-Kähler’ was introduced by Calabi [8, 9], who gave the first non-trivial examples of these metrics, and highlighted their importance prior to their appearance on symplectic quotients and on moduli spaces of anti-self-dual connections [23, 26, 27, 31]. Although our ‘hyper-Kähler constraint’ $\Phi = 0$ is ultimately derived from the Atiyah-Singer Index Theorem, it is in some ways an independent result and may be expected to have a more direct proof. Quite why it respects functorial properties of hyper-Kähler manifolds is briefly explained in Section 3, with reference to the now standard examples of Beauville [4]; more details are in [40]. The remainder of Section 3 is of a different character, as we discuss the topology of a selection of topical examples which fall outside the scope of the title, but which nonetheless are suggestive of a more extensive theory linking geometrical structures and cohomology.

1. Background and main results

Throughout, M denotes a compact oriented manifold of real dimension d . Recall that M is *Kähler* if it possesses both a Riemannian metric g and an orthogonal complex structure J for which the resulting 2-form

$$\omega(X, Y) = g(JX, Y), \quad X, Y \in TM, \quad (4)$$

is closed. We shall always denote the complex dimension of M by n , so that $d = 2n$. We denote the k th Chern class of (the holomorphic tangent bundle of) M by \mathbf{c}_k , and we shall generally regard it as an element of $H^{2k}(M, \mathbb{R})$. In particular, the first Chern class \mathbf{c}_1 of M is represented by $1/2\pi$ times the Ricci tensor (converted into a 2-form by J in analogy to (4)). If \mathbf{c}_1 vanishes over \mathbb{R} , then Yau's proof of the Calabi conjecture implies that M admits a Kähler metric with zero Ricci tensor [44, 6]. This implies that the holonomy group $\text{Hol}(M)$ of the Levi Civita connection is contained in the subgroup $\text{SU}(n)$ of $\text{SO}(d)$.

For example, a quartic hypersurface in \mathbb{CP}^3 is a K3 surface which is manifestly Kähler and has $\mathbf{c}_1 = 0$; it must therefore admit a Kähler ('Calabi-Yau') metric with holonomy group equal to $\text{SU}(2)$. The latter is best identified with the group $\text{Sp}(1)$ of unit quaternions. More generally, the Cheeger-Gromoll splitting theorem implies that a compact Kähler manifold with $\mathbf{c}_1 = 0$ has a finite holomorphic covering of the form

$$T^{2k} \times X_1 \times \cdots \times X_r \times Y_1 \times \cdots \times Y_s, \quad (5)$$

where $T^{2k} \cong \mathbb{C}^k / \mathbb{Z}^{2k}$ is a complex torus, X_i and Y_j are simply-connected, and

$$\text{Hol}(X_i) = \text{SU}(n_i), \quad \text{Hol}(Y_j) = \text{Sp}(m_j)$$

with $m_j \geq 1$ and $n_i \geq 3$. Further details may be found in [4, 6, 7, 28].

A Riemannian manifold M of real dimension $d = 4m$ which, like the Y_j or their products, has $\text{Hol}(M) \subseteq \text{Sp}(m)$ is said to be *hyper-Kähler*. Such a manifold can be characterized by the existence of a triple of orthogonal almost complex structures J_1, J_2, J_3 satisfying the algebraic condition $J_1 J_2 = J_3 = -J_2 J_1$ and such that the associated 2-forms $\omega_1, \omega_2, \omega_3$ defined by (4) are all closed. The latter condition (in contrast to the case of a single almost complex structure) implies that each J_i gives rise to a complex, and therefore Kähler, structure [23, 37]. A hyper-Kähler manifold M actually possesses a continuous family of complex structures, namely $\sum_{i=1}^3 a_i J_i$ with $\sum_{i=1}^3 (a_i)^2 = 1$, parameterized by S^2 or \mathbb{CP}^1 . The resulting deformation or twistor space $M \times \mathbb{CP}^1$ was emphasized from the start in Calabi's study [9]. A hyper-Kähler manifold M is irreducible if and only if $\text{Hol}(M) = \text{Sp}(m)$; in this case $h^{2,0} = 1$. By contrast, the manifolds X_i in (5) have $h^{2,0} = 0$.

We shall denote the Poincaré polynomial of the compact oriented d -manifold M by $P(M; t)$, or by $P(t)$ if the latter causes no confusion. Thus, if the Betti numbers of M are denoted by b_j , $0 \leq j \leq d$, then

$$P(t) = \sum_{j=0}^d b_j t^j = \sum_{j=0}^d b_j t^{d-j}. \quad (6)$$

The alternating sum $P(-1)$ equals the Euler characteristic $e(M)$; our first main result concerns an analogous expression.

Theorem 1. *Let M be a compact oriented manifold of real dimension $d = 4m$. Set*

$$\Phi(M) = 6P''(-1) + \frac{1}{2}d(5 - 3d)P(-1).$$

If M has a hyper-Kähler metric then $\Phi(M) = 0$.

If $m = 1$ then $\Phi(M)$ equals twice the right-hand side of (1); for $m = 2$ we obtain

$$\frac{1}{4}\Phi(M) = 46 - 25b_1 + 10b_2 - b_3 - b_4. \quad (7)$$

A proof of Theorem 1 based upon index theory for the Dirac operator was sketched in [38]. Below, we shall derive it from a corresponding result for almost complex manifolds, Theorem 2 below, by representing Φ as a suitable characteristic number.

There is an elementary reason why the first derivative $P'(-1)$ is not needed in the definition of $\Phi(M)$. For from (6),

$$P'(-1) = - \sum_{j=1}^d (-1)^j j b_j = - \sum_{j=1}^d (-1)^j (d - j) b_j,$$

and adding the last two members,

$$2P'(-1) = -dP(-1). \quad (8)$$

Using this equation, Theorem 1 may be expressed in the equivalent form

$$m e(M) = 6 \sum_{j=0}^{2m-1} (-1)^j (2m - j)^2 b_j. \quad (9)$$

It is well known that on a Kähler manifold, $b_j \equiv 0 \pmod{2}$ if j is odd; similarly on a hyper-Kähler manifold, $b_j \equiv 0 \pmod{4}$ if j is odd [43, 14]. The next result combines this fact with (9).

Corollary. *Let M be a compact hyper-Kähler $4m$ -manifold. Then $m e(M) \equiv 0 \pmod{24}$, and in particular $e(M) \equiv 0 \pmod{2}$ if $m \not\equiv 0 \pmod{8}$.*

Now let M be a compact almost complex manifold of real dimension $2n$. The choice of an almost Hermitian metric on M enables one to define the formal adjoint $\bar{\partial}^* = - * \bar{\partial} *$ of the $\bar{\partial}$ operator. There is then an elliptic differential operator

$$\bigoplus_{q \text{ even}} \Omega^{p,q} \xrightarrow{\bar{\partial} + \bar{\partial}^*} \bigoplus_{q \text{ odd}} \Omega^{p,q},$$

whose index is denoted by χ^p in the notation of [20]. The next results are valid in this general setting, although when the almost complex structure on M is integrable, χ^p is more conveniently defined as $\sum_{q=0}^n (-1)^q h^{p,q}$, where $h^{p,q}$ is the dimension of the corresponding Dolbeault cohomology space or equivalently of the Čech space $H^q(M, \mathcal{O}(\wedge^p T^*))$.

In all cases there is the ‘Serre duality’ relation

$$\chi^{n-p} = (-1)^n \chi^p, \quad (10)$$

and we set

$$\chi(t) = \sum_{p=0}^n \chi^p t^p = (-1)^n \sum_{p=0}^n \chi^{n-p} t^p, \quad (11)$$

which is often denoted by χ_t . For example,

$$\chi(-1) = \sum_{p=0}^n (-1)^p \chi^p = \sum_{j=0}^{2n} (-1)^j b_j \quad (12)$$

coincides with $P(-1) = e(M)$. As we shall explain in the next section, the well-known formula

$$\langle \mathbf{c}_n, [M] \rangle = \chi(-1) \quad (13)$$

may be regarded as the first of a sequence expressing the coefficients of the polynomial $\chi(-1-t)$ (which is in some ways more natural than $\chi(t)$) in terms of Chern numbers. The quadratic term yields

Theorem 2. *Let M be a compact almost complex manifold of real dimension $2n$. Then*

$$\langle \mathbf{c}_1 \mathbf{c}_{n-1}, [M] \rangle = 6\chi''(-1) + \frac{1}{2}n(5-3n)\chi(-1).$$

An equivalent version of the formula can be found at the end of [32]. If the complex dimension n is odd then (12) and (13) imply immediately that $e(M)$ is even. The following are slightly less obvious consequences of Theorem 2 implicit in the theory of SU cobordism described in [41].

Corollary. *Let M be a compact almost complex manifold of real dimension $2n$ with $\mathbf{c}_1 = 0$. Then (i) $n e(M) \equiv 0 \pmod{3}$, and (ii) if $n \equiv 2 \pmod{4}$ then $e(M) \equiv 0 \pmod{2}$.*

As regards (i), which also follows from [21], we remark that when $n = 3$ there are some familiar manifolds which admit non-integrable almost complex structures, for instance S^6 and \mathbb{CP}^3 . If M is simply-connected and $\mathbf{c}_1 = 0$ then the structure group of M reduces to $\mathrm{SU}(n)$ and lifts to $\mathrm{Spin}(2n)$, and (ii) also follows from Ochanine's theorem [34]. The latter states that a compact oriented smooth spin manifold of real dimension $d \equiv 4 \pmod{8}$ has signature divisible by 16 (and therefore $e(M)$ divisible by 2). Other divisibility properties of the Chern numbers appear in the next section, and related results on Chern (or rather Segre) numbers can also be found in [1] and references therein.

2. Proofs and generalizations

In this section, T denotes the holomorphic tangent bundle of an almost complex manifold of real dimension $2n$. We give the total Chern class of T a formal factorization

$$\mathbf{c}(T) = \prod_{i=1}^n (1 + x_i),$$

so that the Chern classes of T may be regarded as the elementary symmetric polynomials in the symbols x_1, \dots, x_n . The Chern character of T is then given by

$$\mathbf{ch}(T) = \sum_{i=1}^n e^{x_i} = n + \mathbf{s}_1 + \frac{1}{2!} \mathbf{s}_2 + \frac{1}{3!} \mathbf{s}_3 + \dots \quad (14)$$

where $\mathbf{s}_k = \sum_{i=1}^n x_i^k$. We shall in fact need $\mathbf{ch}(\bigwedge^p T^*)$, which is formed by replacing the n elements x_i by the $\binom{n}{p}$ elements $-(x_{i_1} + x_{i_2} + \dots + x_{i_p})$, $i_1 < i_2 < \dots < i_p$, which are the weights of the representation defining $\bigwedge^p T^*$. Recall also that the Todd class of T is given by

$$\mathbf{td}(T) = \prod_{i=1}^n \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2} \mathbf{c}_1 + \frac{1}{12} (\mathbf{c}_1^2 + \mathbf{c}_2) + \frac{1}{24} \mathbf{c}_1 \mathbf{c}_3 + \dots$$

With the above preliminaries, the general form of the Hirzebruch-Riemann-Roch theorem [3, 20] implies, making use of (11), that

$$\begin{aligned} \chi(t) &= (-1)^n \sum_{p=0}^n t^p \langle \mathbf{ch}(\bigwedge^{n-p} T^*) \mathbf{td}(T), [M] \rangle. \\ &= (-1)^n \langle \mathbf{ch}(V(t)) \mathbf{td}(T), [M] \rangle, \end{aligned} \quad (15)$$

where

$$V(t) = \sum_{p=0}^n t^p \bigwedge^{n-p} T^* \quad (16)$$

is regarded as an element of $K(M)[t]$. Using the exterior power operation of K-theory, we may write

$$V(-1) = \bigwedge^n (T^* - \underline{\mathbb{C}}),$$

where $\underline{\mathbb{C}}$ denotes a trivial line bundle. Moreover, (16) may be differentiated with respect to t to obtain analogous expressions

$$\begin{aligned} V'(-1) &= \bigwedge^{n-1} (T^* - \underline{\mathbb{C}}^2), \\ V''(-1) &= 2 \bigwedge^{n-2} (T^* - \underline{\mathbb{C}}^3), \end{aligned}$$

where $\underline{\mathbb{C}}^k$ denotes a trivial line bundle with fibre \mathbb{C}^k .

Suppose for a moment that T^* contains $\underline{\mathbb{C}}^3$ as a subbundle, so that $T^* - \underline{\mathbb{C}}^3$ is a genuine complex vector bundle of rank $n - 3$. This effectively corresponds to the case in which $\mathbf{c}_k = 0$ for $k > n - 3$. Then $V''(-1)$ is zero, merely by virtue of its dimension. Accordingly, we may deduce that in general $\mathbf{ch}(V''(-1))$ belongs to the ideal $\langle \mathbf{c}_{n-2}, \mathbf{c}_{n-1}, \mathbf{c}_n \rangle$ generated by the ‘top three’ Chern classes, and the proof of Theorem 2 is completed by the more precise

Lemma. $(-1)^n \mathbf{ch}(V''(-1)) \mathbf{td}(T) = 2\mathbf{c}_{n-2} + (n-1)\mathbf{c}_{n-1} + \frac{1}{12} (2\mathbf{c}_1 \mathbf{c}_{n-1} + n(3n-5)\mathbf{c}_n).$

This equation is derived from the formal factorization

$$(-1)^n \mathbf{ch}(V(-1-t)) \mathbf{td}(T) = \prod_{i=1}^n \left(x_i + t \frac{x_i}{1 - e^{-x_i}} \right). \quad (17)$$

The coefficient of t^2 in (17) is the sum of $\binom{n}{2}$ terms, one of which is

$$\left[1 + \frac{1}{2}(x_1 + x_2) + \frac{1}{12}(x_1^2 + 3x_1x_2 + x_2^2)\right]x_3x_4 \cdots x_n. \quad (18)$$

On the other hand, $\mathbf{c}_1\mathbf{c}_{n-1}$ gives rise to a sum of n terms, one of which is

$$(x_1 + \cdots + x_n)x_2 \cdots x_n = x_1x_2 \cdots x_n + x_2^2x_3 \cdots x_n + \cdots \quad (19)$$

The proof of the lemma is completed from a comparison of (18) and (19).

We may rewrite (15) in the form

$$\chi(-1-t) = \langle \mathbf{K}_n^\bullet(t), [M] \rangle,$$

or equivalently

$$\chi^{(k)}(-1) = (-1)^k k! \langle \mathbf{K}_{n,k}^\bullet, [M] \rangle, \quad (20)$$

where

$$\mathbf{K}_n^\bullet(t) = \mathbf{K}_{n,0}^\bullet + \mathbf{K}_{n,1}^\bullet t + \mathbf{K}_{n,2}^\bullet t^2 + \cdots$$

denotes the component of either side of (17) in $H^{2n}(M, \mathbb{R})$ (we use the notation of [40] in which the bullet indicates a class of top degree). A generalization of the above lemma asserts that both $\mathbf{K}_{n,2k}^\bullet$ and $\mathbf{K}_{n,2k+1}^\bullet$ belong to

$$H^{2n}(M, \mathbb{R}) \cap \langle \mathbf{c}_{n-2k+1}, \mathbf{c}_{n-2k+2}, \dots, \mathbf{c}_n \rangle,$$

with the exception of $\mathbf{K}_{n,0}^\bullet$ which equals \mathbf{c}_n . In fact $\mathbf{K}_{n,2k+1}^\bullet$ is a linear combination of $\mathbf{K}_{n,2j}^\bullet$ for $0 \leq j \leq k$, and we have the following explicit formulae.

Lemma.

$$\begin{aligned} 2^2 3 \mathbf{K}_{n,2}^\bullet &= \mathbf{c}_1 \mathbf{c}_{n-1} + \frac{1}{2} n(3n-5) \mathbf{c}_n, \\ 2^4 3^2 5 \mathbf{K}_{n,4}^\bullet &= [-\mathbf{c}_1^3 + 3\mathbf{c}_1 \mathbf{c}_2 - 3\mathbf{c}_3] \mathbf{c}_{n-3} + [\mathbf{c}_1^2 + 3\mathbf{c}_2] \mathbf{c}_{n-2} + \frac{1}{2} [15n^2 - 85n + 108] \mathbf{c}_1 \mathbf{c}_{n-1} \\ &\quad + \frac{1}{8} n [15n^3 - 150n^2 + 485n - 502] \mathbf{c}_n \\ 2^5 3^3 5^1 7 \mathbf{K}_{n,6}^\bullet &= [\mathbf{c}_1^5 - 5\mathbf{c}_1^3 \mathbf{c}_2 + 5\mathbf{c}_1 \mathbf{c}_2^2 + 5\mathbf{c}_1^2 \mathbf{c}_3 - 5\mathbf{c}_2 \mathbf{c}_3 - 5\mathbf{c}_1 \mathbf{c}_4 + 5\mathbf{c}_5] \mathbf{c}_{n-5} \\ &\quad + \frac{1}{2} [-2\mathbf{c}_1^4 + \mathbf{c}_1^2 \mathbf{c}_2 + 10\mathbf{c}_2^2 - \mathbf{c}_1 \mathbf{c}_3 - 20\mathbf{c}_4] \mathbf{c}_{n-4} + \frac{1}{4} [-(21n^2 - 203n + 472) \mathbf{c}_1^3 \\ &\quad + (63n^2 - 609n + 1430) \mathbf{c}_1 \mathbf{c}_2 - (63n^2 - 609n + 1388) \mathbf{c}_3] \mathbf{c}_{n-3} \\ &\quad + \frac{1}{4} [(21n^2 - 203n + 472) \mathbf{c}_1^2 + (63n^2 - 609n + 1408) \mathbf{c}_2] \mathbf{c}_{n-2} \\ &\quad + \frac{1}{16} [105n^4 - 1890n^3 + 12131n^2 - 32242n + 28800] \mathbf{c}_1 \mathbf{c}_{n-1} \\ &\quad + \frac{1}{96} n [63n^5 - 1575n^4 + 15435n^3 - 73801n^2 + 171150n - 152696] \mathbf{c}_n. \end{aligned}$$

We see from (20) that, as k increases, $\chi^{(k)}(-1)$ involves progressively more Chern numbers, and only if $k = n$ is even do we obtain an expression

$$\chi^{(n)}(-1) = n! \chi^n = (-1)^n n! \chi^0 \quad (21)$$

(a multiple of the Todd genus) in which the term \mathbf{c}_1^n appears. Taking $k = 4$ in (20),

Theorem 3. *Let M be a compact almost complex manifold of real dimension $2n$ with $\mathbf{c}_1 = 0$. Then*

$$\langle \mathbf{c}_2 \mathbf{c}_{n-2} - \mathbf{c}_3 \mathbf{c}_{n-3}, [M] \rangle = 10\chi^{(iv)}(-1) - \frac{1}{24}n(15n^3 - 150n^2 + 485n - 502)\chi(-1).$$

For example, when $n = 4$ and $\mathbf{c}_1 = 0$, this is equivalent to the fact that the top Todd class reduces to $(3\mathbf{c}_2^2 - \mathbf{c}_4)/720$.

Corollary. *Let M be a compact almost complex manifold of real dimension $2n$ with $\mathbf{c}_1 = 0$. Then $2n\mathbf{c}_n + \mathbf{c}_2\mathbf{c}_{n-2} - \mathbf{c}_3\mathbf{c}_{n-3} \equiv 0 \pmod{5}$.*

More generally, on any almost complex $2n$ -manifold, the indicated summands of Newton's formula

$$\underbrace{n\mathbf{c}_n - \mathbf{s}_1\mathbf{c}_{n-1} + \cdots - \mathbf{s}_{k-1}\mathbf{c}_{n-k+1}} + \underbrace{\mathbf{s}_k\mathbf{c}_{n-k} + \cdots + (-1)^n\mathbf{s}_n}_{=0} = 0$$

are individually zero modulo $k+1$ if $k+1 \geq 3$ is prime [21, 40].

Let us now turn attention to the derivation of Theorem 1 from Theorem 2. First suppose that M is a compact hyper-Kähler manifold of real dimension $d = 2n = 4m$. If we fix a compatible complex structure J_1 then the closed 2-form $\eta = \omega_2 + i\omega_3$ is holomorphic, and η^m is nowhere zero. Wedging by η^{m-p} is known to induce an isomorphism $H^{p,q} \rightarrow H^{n-p,q}$ between the appropriate Dolbeault cohomology spaces, which implies that the Hodge numbers of a hyper-Kähler manifold are invariant by the mirror symmetry (3); this was proved by Fujiki [14]. Also, the $(n,0)$ -form η^m trivializes the canonical bundle of M , so that $\mathbf{c}_1 = 0$.

The Betti and Hodge numbers of the Kähler manifold M of real dimension $2n$ are related by the formula $P(t) = H(t, t)$, where

$$H(x, y) = \sum_{p,q=0}^n h^{p,q} x^p y^q$$

is the so-called Hodge polynomial. Because $H(x, y)$ is symmetric in x and y , we have

$$\begin{aligned} P''(-1) &= H_{xx}(-1, -1) + 2H_{xy}(-1, -1) + H_{yy}(-1, -1) \\ &= 2(H_{xx}(-1, -1) + H_{xy}(-1, -1)). \end{aligned}$$

Firstly, $H(t, -1) = \chi(t)$, so

$$H_{xx}(-1, -1) = \chi''(-1).$$

Secondly,

$$\begin{aligned} H_{xy}(-1, -1) &= \sum_{p,q=0}^n (-1)^{p+q} pq h^{p,q} \\ &= -\frac{1}{2}n\chi'(-1) + \frac{1}{2} \sum_{p,q=0}^n (-1)^{p+q} [pq - (n-p)q] h^{p,q} \\ &= \frac{1}{4}n^2\chi(-1) + \frac{1}{2} \sum_{p,q=0}^n (-1)^{p+q} pq (h^{p,q} - h^{n-p,q}). \end{aligned}$$

The last equality uses an analogue of (8) that follows from (10).

Combining the above equations, we obtain

Lemma. $\Phi(M) = 2[6\chi''(-1) + \frac{1}{2}n(5-3n)\chi(-1)] + 6 \sum_{p,q=0}^n (-1)^{p+q} pq(h^{p,q} - h^{n-p,q}).$

From Theorem 2 we deduce that when $\mathbf{c}_1 \mathbf{c}_{n-1} = 0$, the symmetry (3) reverses the sign of Φ . The latter is therefore zero in the hyper-Kähler case.

3. Examples and remarks

If M and N are both hyper-Kähler, then their Riemannian product $M \times N$ admits an obvious hyper-Kähler structure. We must therefore understand why the constraint of Theorem 1 is preserved by the process of taking products.

First suppose that M is an almost complex manifold of real dimension $2n$ with $e(M) \neq 0$. If we set

$$\gamma(M) = \frac{\langle \mathbf{c}_1 \mathbf{c}_{n-1}, [M] \rangle}{\langle \mathbf{c}_n, [M] \rangle},$$

the identity $\mathbf{c}(M \times N) = \mathbf{c}(M)\mathbf{c}(N)$ for the total Chern class implies that

$$\gamma(M \times N) = \gamma(M) + \gamma(N). \quad (22)$$

It follows from Theorem 2 that the quantity

$$\frac{6\chi''(-1) + \frac{1}{2}n(5-3n)\chi(-1)}{\chi(-1)},$$

and therefore

$$\psi = \frac{4\chi''(-1)}{\chi(-1)} - n^2,$$

also satisfies (22) in place of γ . When M has even real dimension d , the same must be true of

$$\phi = \frac{4P''(-1)}{P(-1)} - d^2 \quad (23)$$

(also defined in [38]), since the coefficients of $P(t)$ satisfy (10) with d in place of n .

In fact the additivity of ϕ is as elementary as that of γ , and can be deduced from the observation that ϕ is proportional to the coefficient of t^2 in the formal power series

$$\log P(-1+t) = \log e(M) - \frac{1}{2}dt + \frac{1}{8}\phi t^2 + \frac{1}{24}(3\phi + 2d)t^3 + \dots$$

and similar remarks apply to ψ . Referring to Theorem 1, and still assuming that $e(M) \neq 0$, we have $\Phi(M) = (3\phi(M) + 5d)e(M)/2$. Thus, $\Phi(M)$ is zero if and only if

$$\frac{\phi(M)}{\dim(M)} = -\frac{5}{3},$$

and we may think of the right-hand side as a ‘coupling constant’ for hyper-Kähler manifolds.

Hilbert schemes of points. Let S be a compact complex surface, and let $S^{(m)}$ denote its m -fold symmetric product obtained by quotienting the Cartesian product by the group of permutations. There is a resolution

$$\varepsilon: S^{[m]} \longrightarrow S^{(m)}$$

in which $S^{[m]}$ is the Hilbert scheme of closed 0-dimensional subschemes of length m on S , and is a smooth complex $2m$ -dimensional manifold. Each non-trivial fibre $\varepsilon^{-1}(x)$ is a product $(V_2)^{\alpha_2} \times \cdots \times (V_m)^{\alpha_m}$, where $V_i = \text{Hilb}^i(\mathbb{C}[x, y])$ is the scheme that parameterizes ideals in $\mathbb{C}[x, y]$ of colength i , and α_i denotes the number of i -tuples of points that have coalesced in $x \in S^{(m)}$ [12, 16].

Following an example of Fujiki [13], Beauville has proved [4] that if S has a holomorphic symplectic structure then so does $S^{[m]}$ for all $m \geq 2$; its holomorphic 2-form is induced from a natural one on $S^{(m)}$. It follows that if S admits a hyper-Kähler metric then so does $S^{[m]}$ for any $m \geq 2$, and we may apply this construction when S is a K3 surface or a torus. If $S = T$ is a torus, then $T^{[m]}$ is not locally irreducible and has a 4^m -fold covering of the form (5) with $k = 2$ and unique non-flat factor Y_1 of real dimension $4m - 4$. The latter can also be viewed as a submanifold of $T^{[m]}$ and is denoted K_{m-1} in [4] and $A^{[m]}$ in [17]; when $m = 2$ it is merely the Kummer surface associated to T .

For any manifold S , the Betti numbers of $S^{(m)}$ were computed by Macdonald [30]. When S has real dimension 4, we have

$$\sum_{m \geq 0} P(S^{(m)}; t) x^m = \frac{(1 + tx)^{b_1} (1 + t^3 x)^{b_3}}{(1 - x)^{b_0} (1 - t^2 x)^{b_2} (1 - t^4 x)^{b_4}}. \quad (24)$$

The form of the right-hand side indicates its generalization to higher dimensions, though in the present context we have $b_1 = b_3$ and we may assume that $b_0 = 1 = b_4$. Deeper results of Göttsche and Soergel [16, 17] give the Betti numbers of $S^{[m]}$, at least when S is a projective surface, and building on (24), we have

$$P(S^{[m]}; t) = \sum_{\alpha} \prod_{i=1}^m P(S^{(\alpha_i)}; t) t^{2m-2\alpha_i}. \quad (25)$$

The sum is over all partitions α of m , each of which is uniquely determined by m non-negative integers $\alpha_1, \dots, \alpha_m$ with the property that $m = \sum_{i=1}^m i\alpha_i$.

Given that the Betti numbers of $K^{[2]}$ and K_2 both satisfy the constraint $\Phi = 0$ given in (7), one may deduce independently of (25) that

$$\begin{aligned} P(K^{[2]}; t) &= 1 + 23t^2 + 276t^4 + 23t^6 + t^8, \\ P(K_2; t) &= 1 + 7t^2 + 8t^3 + 108t^4 + 8t^5 + 7t^6 + t^8, \end{aligned}$$

and the Euler characteristics are

$$\begin{aligned} e(K^{[2]}) &= 324, \\ e(K_2) &= 108. \end{aligned}$$

One needs (25) and a related formula in [17] to treat the higher-dimensional cases, and it is amusing to note that

$$\begin{aligned} e(K^{[8]}) &= 30178575, \\ e(K_8) &= 9477 \end{aligned}$$

are both odd. As predicted by Theorem 1, we have

$$\Phi(K^{[m]}) = 0 = \Phi(K_m) \quad (26)$$

for all $m \geq 1$; by contrast, $\Phi(K^{(m)}) = 24m(m-1)/25$. Underlying (26) is the fact that for any projective surface S with $e(S) \neq 0$, one has

$$\begin{aligned}\phi(S^{[m]}) &= m\phi(S), \\ \psi(S^{[m]}) &= m\psi(S).\end{aligned}\tag{27}$$

In particular, the second formula is encoded in the Hodge polynomial of $S^{[m]}$ computed in [17] and [10]; further details can be found in [40]. We emphasize that the equations (27) have taken us outside the realm of manifolds with $\mathbf{c}_1 = 0$, and we shall now take the opportunity to move further afield.

Other holonomy groups. Let M be a compact oriented manifold of real dimension $d = 2n$. The constituents $P(M; -1)$ and $P''(M; -1)$ of $\Phi(M)$ are both zero if and only if $(1+t)^4$ is a factor of $P(M; t)$. This is certainly the case if $k \geq 2$ in the decomposition (5). Now suppose that $M = N \times S^1$, so that

$$P(M; t) = P(N; t)(1+t),$$

and the Euler characteristic $P(N; -1)$ is zero. Then $\Phi(M) = 0$ if and only if $P(N; t)$ is divisible by $(1+t)^3$, which is equivalent to the equation $P'(N; -1) = 0$. Whilst these observations are elementary and of little interest in the case in which M is hyper-Kähler, it seems that when $d = 8$ both $\Phi(M)$ and

$$P'(N, -1) = -b_3 + 3b_2 - 5b_1 + 7\tag{28}$$

are natural quantities to consider in the context of other holonomy reductions. Now, if M is an irreducible Riemannian d -manifold with zero Ricci tensor whose holonomy group $\text{Hol}(M)$ is a proper subgroup of $\text{SO}(d)$, then Berger's theorem [5, 6] implies that exactly one of the following situations occurs: (i) M is Kähler and $\text{Hol}(M) \subseteq \text{SU}(d/2)$, (ii) $d = 7$ and $\text{Hol}(M) \cong \text{G}_2$, or (iii) $d = 8$ and $\text{Hol}(M) \cong \text{Spin}(7)$.

Various examples of compact 7-manifolds of type (ii) have now been described by Joyce [24] by smoothing orbifolds of the form T^7/Γ , where Γ is a finite group acting on T^7 preserving the flat G_2 -structure. The latter is characterized by an invariant 3-form $\varphi = \sum_{i=1}^7 \varphi_i$ on \mathbb{R}^7 which is the sum of simple 3-forms φ_i , $1 \leq i \leq 7$. The first example announced by Joyce actually had Betti numbers

$$b_1 = 0, \quad b_2 = 12, \quad b_3 = 43,\tag{29}$$

so that (28) vanishes. In this case, $\Gamma \cong (C_2)^3$ is an abelian group of affine transformations preserving φ , and each element of order 2 acts trivially on the 3-dimensional subspace of \mathbb{R}^3 determined by exactly one of the φ_i ; moreover T^7/Γ has Betti numbers $b_1 = 0 = b_2$, $b_3 = 7$ so that $P'(T^7/\Gamma; -1) = 0$. Topologically, the smoothing process replaces each of 12 singular 3-tori by $T^3 \times S^2$; each replacement adds 1 to the second Betti number and 3 to the third, thereby preserving (28). Other examples turned out to be less respectful of (28), although at the orbifold level for any Γ it is necessarily the case that $P'(T^7/\Gamma; -1) \geq 0$.

If N has type (ii) above, then the holonomy of the product metric on $M = N \times S^1$ is certainly a subgroup of $\text{Spin}(7)$. Now, manifolds with holonomy contained in $\text{Spin}(7)$ generalize those with holonomy equal to $\text{SU}(4)$, and at the same time the two structure groups have much in common. For example, the equation $\langle 4\mathbf{p}_2 - \mathbf{p}_1^2, [M] \rangle = 8e(M)$, which is valid for any almost complex 8-manifold M with $\mathbf{c}_1 = 0$ (since then $\mathbf{p}_1 = -2\mathbf{c}_2$ and $\mathbf{p}_2 = 2\mathbf{c}_4 + \mathbf{c}_2^2$), is in fact also satisfied in the $\text{Spin}(7)$ case, essentially because $\text{SU}(4)$ and $\text{Spin}(7)$ share a maximal torus [18]. Given that $\text{SU}(4)$ mirror

symmetry changes the sign of Φ , in addition to fixing b_3 and $2b_2 + b_4$, it is curious to see what can be said about these quantities in the $\text{Spin}(7)$ case. For example, according to Joyce, there is a compact 8-manifold with holonomy equal to $\text{Spin}(7)$ whose Betti numbers

$$b_1 = 0, \quad b_2 = 12, \quad b_3 = 16, \quad b_4 = 150 \quad (30)$$

satisfy the constraint $\Phi = 0$ (see (7)). In this case, some standard index theory yields in addition

$$b_4^- = 3b_2 + 7 = 43$$

(cf. (29)), where b_4^- is the dimension of the space of harmonic anti-self-dual 4-forms α (i.e., those satisfying $*\alpha = -\alpha$).

The index theory calculations in Section 2 grew out of similar ones for the class of quaternion-Kähler manifolds, which are characterized by the condition $\text{Hol}(M) \subseteq \text{Sp}(d/4)\text{Sp}(1)$. Unless $\text{Hol}(M) \subseteq \text{Sp}(d/4) \subseteq \text{SU}(d/2)$ (the hyper-Kähler case), there is no compatible Kähler structure and the Ricci tensor is non-zero. Nevertheless the Betti numbers of a compact quaternion-Kähler manifold M of positive scalar curvature are also subject to a linear relation analogous to Theorem 1. It is known that the odd Betti numbers of M are zero, and that if $d = 4m$ then the integers

$$\beta_{2k} = b_{2k}(M) - b_{2k-4}(M), \quad 0 \leq k \leq m,$$

(where $b_j = 0$ for $j < 0$) are non-negative [14]. These ‘primitive Betti numbers’ are subject to their own remarkable constraint

$$\sum_{k=1}^m k(m+1-k)(m+1-2k)\beta_{2k} = 0, \quad (31)$$

which is an equivalent but neater form of the relation on the ordinary Betti numbers of M that appears in [29, 39].

The character of (31) no doubt reflects its validity for certain symmetric spaces which are the obvious (and conceivably only) candidates for M . For instance, if $m = 7$ then (31) becomes

$$7\beta_2 + 8\beta_4 + 5\beta_6 = 5\beta_{10} + 8\beta_{12} + 7\beta_{14},$$

and is satisfied not only by the projective space \mathbb{HP}^7 ($\beta_{2k} \equiv 0$), the Grassmannians $\mathbb{G}r_2(\mathbb{C}^9)$ ($\beta_{2k} \equiv 1$) and $\widetilde{\mathbb{G}r}_4(\mathbb{R}^{11})$ ($\beta_{4k} \equiv 1$, $\beta_{4k-2} \equiv 0$), but also in a striking way by the exceptional space $F_4/\text{Sp}(3)\text{Sp}(1)$ ($\beta_{2k} = 0$ for $k \neq 4$, $\beta_8 = 1$). The arithmetic is more intriguing for the E -type quaternion-Kähler symmetric spaces, since their primitive Betti numbers satisfy the equation $\beta_{2m-2k} = \beta_{2k}$ which is not obviously compatible with the $k \leftrightarrow m+1-k$ invariance of (31)! The resulting theory is closely related to that of the signature and other elliptic genera of homogeneous spaces, which is the subject of [22].

A natural notion of self-dual connection for vector bundles over hyper-Kähler and quaternion-Kähler manifolds has long been known. This theory extends in a natural way to any Riemannian manifold with reduced holonomy, by requiring that the curvature of the connection take values in a simple summand of the holonomy algebra. For each such algebra, the resulting problem has an well-defined elliptic complex whose first cohomology group parameterizes infinitesimal deformations of the connection [36]. Investigation of the corresponding moduli spaces can be expected to provide further interplay between index theory and the topology of the manifolds discussed above.

An example with $\mathbf{c}_1 > 0$. We conclude with an example of a Kähler manifold with ample anti-canonical bundle. Namely, let \mathcal{M}_g denote the moduli space of stable rank 2 vector bundles V over a Riemann surface of genus $g \geq 2$ with $\bigwedge^2 V$ isomorphic to a fixed line bundle of degree one. Then \mathcal{M}_g is a smooth Fano manifold of complex dimension $n = 3g - 3$ and index 2, its first Chern class \mathbf{c}_1 being twice a positive integral class [35]. The Betti numbers of \mathcal{M}_g were first given by Newstead [33], and subsequently determined by a variety of methods [2, 11, 19]. In particular, Atiyah and Bott include a comparison of their own equivariant Morse theory methods with the number-theoretic approaches, and also explain how \mathcal{M}_g arises from an infinite-dimensional symplectic quotient construction. The Poincaré polynomial of \mathcal{M}_g is

$$P(t) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)} = (1+t)^{2g-2} \sum_{i=0}^{g-1} (1-t+t^2)^{2i} t^{2g-2-2i}.$$

In particular, for all g ,

$$b_2(\mathcal{M}_g) = 1, \quad b_3(\mathcal{M}_g) = 2g,$$

the sum $P(1)$ of the Betti numbers equals $2^{2g-2}g$, and the Euler characteristic $P(-1)$ is zero.

Because of the property $P^{(i)}(-1) = 0$, $i \leq 2g - 1$, the simply-connected manifold \mathcal{M}_g obviously satisfies the constraint $\Phi(\mathcal{M}_g) = 0$ of Theorem 1. This leads one to consider the polynomial $\chi_t = \chi(t)$ which can in theory be computed from a knowledge of the Chern classes \mathbf{c}_k of \mathcal{M}_g . Newstead and Ramanan made a number of conjectures concerning the characteristic classes of \mathcal{M}_g which have been subsequently proved. In particular, $\mathbf{c}_k = 0$ for $k > 2g - 2$ [16], and the Pontrjagin ring (which is known to be generated solely by \mathbf{p}_1) vanishes in degrees $4g$ and above [25, 42]. We shall illustrate consequences of these facts in the relatively simple case $g = 3$ which is nonetheless indicative of the general situation.

The relations $\mathbf{c}_5 = 0 = \mathbf{c}_6$ on \mathcal{M}_3 imply immediately that

$$\chi(-1+t) = at^4 + bt^5 + ct^6,$$

where $a, b, c \in \mathbb{Z}$. From (21), $c = \chi^0$ is the Todd genus, and equals 1 because $\mathbf{c}_1 > 0$. Furthermore, $a = \chi^{(iv)}(-1)/4!$ is given by

$$\langle \mathbf{K}_{6,4}^\bullet, [\mathcal{M}_3] \rangle = \frac{1}{720} [(-\mathbf{c}_1^3 + 3\mathbf{c}_1\mathbf{c}_2 - 3\mathbf{c}_3)\mathbf{c}_3 + (\mathbf{c}_1^2 + 3\mathbf{c}_2)\mathbf{c}_4],$$

which is seen to equal 4, for example using expressions for the Chern classes in [35]. Consequently,

$$\chi(t) = (1+t)^4((b+5) + (b+2)t + t^2),$$

and from (10) we deduce that $b = -4$. We thereby arrive at a special case of the more general result

$$\chi(t) = (1+t)^{2g-2}(1-t)^{g-1} \tag{32}$$

which was proved in [32], and could also be derived from a knowledge of the Hodge polynomial of \mathcal{M}_g . Given that $b_2(\mathcal{M}_g) = 1$, $e(\mathcal{M}_g) = 0$, and $\mathbf{c}_1 \neq 0$, the vanishing of \mathbf{c}_{3g-2} may be deduced from that of $\chi''(-1)$, although the vanishing of \mathbf{c}_k for $2g-2 < k < 3g-2$ appears to be altogether deeper. Finally, observe that (32) implies that the signature $\chi(1)$ of \mathcal{M}_g is zero for all g , which is consistent with the vanishing of all Pontrjagin numbers.

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